2.5 & 2.6 Review

Related Rates:
1. A 25 ft ladder rests against a vertical wall. If the bottom of the ladder is sliding away from the base of the wall at a rate of 3 ft/sec, how fast is the top of the ladder moving down from the wall when the bottom of the ladder is 7 ft from the base?

\[
\frac{dx}{dt} = 3 \text{ ft/sec} \quad \frac{dy}{dt} \bigg|_{x=7} \]

\[x^2 + y^2 = 25^2\]

\[2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0\]

\[2(7)(3) + 2(24) \frac{dy}{dt} = 0\]

\[\frac{dy}{dt} = -\frac{875}{288} \text{ ft/sec}\]

2. Gas is escaping from a spherical balloon at the rate of 2 ft\(^3\)/min. How fast is the surface area shrinking when the radius is 12 ft?

\[
\frac{dV}{dt} = -2 \text{ ft}^3/\text{min.}
\]

\[V = \frac{4}{3} \pi r^3\]

\[\frac{dV}{dt} = \frac{4}{3} \pi r^2 \frac{dr}{dt}\]

\[-2 = \frac{4}{3} \pi (12)^2 \frac{dr}{dt}\]

\[-\frac{1}{288\pi} = \frac{dr}{dt}\]

\[
\frac{dS}{dt} = \frac{1}{3} \frac{2\pi^2 r^2}{\text{sec}} \approx 3 \text{ ft/sec}
\]

3. Sand falling from a chute forms a conical pile whose altitude is always equal to \(4/3\) the radius of the base. (a) How fast is the volume increasing when the radius of the base is 3 ft and is increasing at the rate of 3 in/min? (b) How fast is the radius increasing when it is 6 ft and the volume is increasing at the rate of 24 ft\(^3\)/min?

\[
V = \frac{1}{3} \pi r^2 h
\]

\[
V = \frac{1}{3} \pi r^2 \left(\frac{4}{3}r\right)
\]

\[
V = \frac{4}{9} \pi r^3
\]

\[
\frac{dV}{dt} = \frac{12}{9} \pi r^2 \frac{dr}{dt}
\]

\[
\frac{dV}{dt} = \frac{4}{3} \pi (3)^2 \cdot \frac{1}{4}
\]

\[
\frac{dV}{dt} = \frac{3\pi}{\text{ft}^3/\text{min}}
\]

\[
\frac{1}{2\pi} \frac{dV}{dt} = \frac{dr}{dt}
\]

\[
24 = \frac{4\pi}{3} (b)^2 \frac{dr}{dt}
\]
4. Two parallel sides of a rectangle are being lengthened at the rate of 2 in/sec, while the other two sides are shortened in such a way that the figure remains a rectangle with constant area $A = 50 \text{ in}^2$. What is the rate of change of the perimeter $P$ when the length of an increasing side is (a) 5 in? (b) 10 in? (c) What are the dimensions when the perimeter ceases to decrease?

See attached Sheet

Implicit Differentiation:

1. Find $\frac{dy}{dx}$ if $\sin(x + y) = 2x$

2. Find $\frac{dy}{dx}$ of $\frac{2x - 5y^2}{4y^3 - x^2} = -x$ at $(1, 1)$

3. Find $\frac{dy}{dx}$ if $x \sin y + y \sin x = \frac{\pi}{2\sqrt{2}}$ at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

See attached Sheets
4) \[
\frac{dx}{dt} = \frac{2un}{\sec}
\]
\[\text{Area} = 50\]

a) we want \(\frac{dP}{dt}\) \(\left|_{x=5, \ y=10}\right.\)

\[P = 2x + 2y\]

\[\frac{dP}{dt} = 2 \frac{dx}{dt} + 2 \frac{dy}{dt}\]

* If \(x=5\), then \(y\) must be 10 since the area is 50.

Next, we need to find \(\frac{dy}{dt}\) using area.

\[50 = xy\]

\[0 = y \frac{dx}{dt} + x \frac{dy}{dt}\]

Product rule: \(u=x, \ v=y\)

\[u'=\frac{dx}{dt}, \ v'=\frac{dy}{dt}\]

\[0 = 10(2) + 5 \frac{dy}{dt}\]

\[-4 \frac{\text{in}}{\text{sec}} = \frac{dy}{dt}\]

\[\frac{dP}{dt} = 2 \frac{dx}{dt} + 2 \frac{dy}{dt}\]

\[\frac{dP}{dt} = 2(2) + 2(-1) = \boxed{-4 \frac{\text{in}}{\text{sec}}}\]

b) \(\frac{dP}{dt}\) \(\left|_{x=10, \ y=5}\right.\)

\[0 = y \frac{dx}{dt} + x \frac{dy}{dt}\]

\[0 = 5(2) + 10 \frac{dy}{dt}\]

\[-1 = \frac{dy}{dt}\]

\[\frac{dP}{dt} = 2(2) + 2(-1) = \boxed{2 \frac{\text{in}}{\text{sec}}}\]

Now, we must find the dimensions when \(\frac{dy}{dt} = -2\).

\[\frac{dt}{dt} = x \frac{dy}{dt} + y \frac{dx}{dt}\]

\[7A = xy = 50\]

\[0 = x(-2) + y(2)\]

\[0 = -2x + 2y\]

\[2x = 2y\]

\[\boxed{x = y}\]

\[\sqrt{2} = \frac{\text{in}}{\text{sec}}\]

We can now find the actual dimensions.

\[x = 5\sqrt{2} \text{ in} \]

\[y = 5\sqrt{2} \text{ in}\]
Implicit Differentiation:

1) Find \( \frac{dy}{dt} \) if \( \sin(x+y) = 2x \)

\[
\cos(x+y) \left(1 + \frac{dy}{dx}\right) = 2
\]

\[
\cos(x+y) + \cos(x+y) \frac{dy}{dx} = 2
\]

\[
\cos(x+y) \frac{dy}{dx} = 2 - \cos(x+y)
\]

\[
\frac{dy}{dx} = \frac{2 - \cos(x+y)}{\cos(x+y)}
\]

2) Find \( \frac{dy}{dx} \) of \( \frac{2x - 5y^2}{4y^3 - x^2} = -x \) at \((1, 1)\)

*Cross multiply first* (you could go straight to quotient rule, but that would be miserable)

\[
2x - 5y^2 = -4xy^3 + x^3
\]

\[
2 - 10y \frac{dy}{dx} = -4y^3 + 3xy^2 \frac{dy}{dx} + 3x^2
\]

\[
2 - 10y \frac{dy}{dx} = -4y^3 - 12xy^2 \frac{dy}{dx} + 3x^2
\]

\[
12xy^2 \frac{dy}{dx} - 10y \frac{dy}{dx} = 3x^2 - 4y^2 - 2
\]

\[
\frac{dy}{dx} \left(12xy^2 - 10y\right) = 3x^2 - 4y^2 - 2
\]

\[
\frac{dy}{dx} = \frac{3x^2 - 4y^2 - 2}{12xy^2 - 10y}
\]

\[
\frac{dy}{dx} \bigg|_{(1, 1)} = \frac{3(1)^2 - 4(1) - 2}{12(1)(1)^2 - 10(1)} = \frac{-3}{2}
\]
Find \( \frac{dy}{dx} \) if \( x \sin y + y \sin x = \frac{\pi}{2x} \) @ \( \left( \frac{\pi}{4}, \frac{\pi}{4} \right) \)

\( u = x \quad u' = 1 \), \( v = \sin y \quad v' = \cos y \frac{dy}{dx} \)

\( u = y \quad u' = \frac{dy}{dx} \), \( v = \sin x \quad v' = \cos x \)

\[
\sin y + x \cos y \frac{dy}{dx} + \sin x \frac{dy}{dx} + y \cos x = 0
\]

\[
x \cos y \frac{dy}{dx} + \sin x \frac{dy}{dx} = -\sin y - y \cos x
\]

\[
\frac{dy}{dx} \left( x \cos y + \sin x \right) = -\sin y - y \cos x
\]

\[
\frac{dy}{dx} = \frac{-\sin y - y \cos x}{x \cos y + \sin x} \quad \text{@} \left( \frac{\pi}{4}, \frac{\pi}{4} \right)
\]

\[
= \frac{-\sin \frac{\pi}{4} - \frac{\pi}{4} \cos \frac{\pi}{4}}{\frac{\pi}{4} \cos \frac{\pi}{4} + \sin \frac{\pi}{4}} \quad \frac{-\sqrt{2} - \frac{\pi}{4} \sqrt{2}}{\frac{\pi}{4} \sqrt{2} + \sqrt{2} \frac{\pi}{4}} = \boxed{-1}
\]